Cohomology

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1 Axioms

The axioms for a cohomology theory are given by *dualising* the homology axioms. In particular just turn all arrows around.

2 Definition

Here we will define cellular / singular cohomology. First form the chain complex (we will be ambiguous about which ones)

$$\cdots \to C_n \xrightarrow{\partial} C_{n-1} \to \cdots$$

then apply $\mathrm{Hom}(-, G)$ to get another "co"chain

$$\cdots \leftarrow \mathrm{Hom}(C_n, G) \xleftarrow{\mathrm{Hom}(\partial, G)} \mathrm{Hom}(C_{n-1}, G) \leftarrow \cdots$$

we then take the co-homology of this co-chain.

If G is am Abelian group then we are homing in the category of Abelian groups. If G is a unital ring then we are homing in the category of G-modules.

Note that this is the level of the chain. That is if Hom commuted with homology we would have the same theory. The fact is that Hom *does not* commute with taking the homology of a chain and therefore the groups of *cohomology* may be different to the groups of *homology*.

For homology the boundary map is

$$R[\sigma_i^n] \to R[\sigma_i^{n-1}]$$
$$\partial(\sigma_i^n) \quad := \quad \sum_i (-1)^i \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}$$

this induces the following map on cochains

$$\delta : \mathrm{Hom}(R[\sigma_i^{n-1}], R) \to \mathrm{Hom}(R[\sigma_i^n], R)$$
$$\delta(\varphi) = \varphi(\partial)$$

or in full

$$\delta(\varphi)(\sigma^n) = \varphi\big(\sum_i (-1)^i \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}\big)$$

A **cocycle** is an element of $\ker \partial$, that is the kernel of the boundary and a coboundary is an element of the image of the boundary map.

3 Relation to Homology

Our chain groups are all $\mathbb{Z}^n$ and hence we need to see what $\mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$ is. We know that $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$ analogously we have that $\mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}) = \mathbb{Z}^n$, with the maps being given by

$$\big((k_1, ..., k_n) \mapsto \sum_i m_i k_i\big) \mapsto (m_1, ..., m_n)$$

Lemma (Hatcher Ex. 43, §2.2)**.** *Any chain complex of finitely generated abelian groups splits as the direct sum of the complexes*

$$0 \to \mathbb{Z} \to 0, \qquad 0 \to \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \to 0$$

Homing this lemma tells us that the dual complexes have have the same homology groups except tosion is shifted up one dimension.

Example (Hom does not commute)**.** *Consider the following complex*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \longrightarrow & 0 \\ & & \shortparallel & & \shortparallel & & \shortparallel & & \shortparallel & & \\ & & C_3 & & C_2 & & C_1 & & C_0 & & \end{array}$$

where n is the map $x \mapsto nx$. Then the dual chain is

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \longrightarrow & 0 \\ & & \shortparallel & & \shortparallel & & \shortparallel & & \shortparallel & & \\ & & C_0^* & & C_1^* & & C_2^* & & C_3^* & & \end{array}$$

Here we can see even though the chain is exactly the same the numbers are different. Lets compute the homology and cohomology.

$$H_n(chain) = \begin{cases} \mathbb{Z}, & n = 3 \\ 0, & n = 2 \\ \mathbb{Z}/2\mathbb{Z}, & n = 1 \\ \mathbb{Z}, & n = 0 \end{cases}$$

$$H_n(co\text{-}chain) = \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n = 1 \\ \mathbb{Z}/2\mathbb{Z}, & n = 2 \\ \mathbb{Z}, & n = 3 \end{cases}$$

We can see that the degree 1 and 2 groups swap, actually the degree 1 group has been "pushed up". THIS ALL FEELS VERY FORMAL, IT FEELS LIKE I SHOULD BE ABLE TO RELABEL THE CHAIN SUCH THAT HOMOLOGY AND COHOMOLGY AGREE...? MAYBE THE DISTINCTION BECOMES MORE PROFOUND FOR GROUPS OTHER THAN $\mathbb{Z}$? something about our modules in the chain complex being free over a non-torsion group...?

We will now make the relationship between them precise.

Lemma. *There is a split SES*

define the h map

$$0 \to \ker h \to H^n(C; G) \xrightarrow{h} \mathrm{Hom}(H_n(C), G) \to 0$$

We can recall the general definition of Ext groups, however in the setting here we only need a base case. Given an abelian group A then there is always a resolution by free abelian groups, an exact sequence, of the form

$$0 \to F_1 \to F_0 \to A \to 0$$

Apply $\mathrm{Hom}(-, G)$ and taking cohomology we get

$$F_1* \leftarrow F_0* \leftarrow A* \leftarrow 0$$

which is still exact. We lost exactness at the first group however (Hom is *left exact, but not right exact*). Thus there may be a non-trivial homology at this point which we denote

$$\mathrm{Ext}^1(A, G).$$

Ext can be computed with the properties

- $\mathrm{Ext}(A \oplus A', G) \cong \mathrm{Ext}(A, G) \oplus \mathrm{Ext}(A', G)$
- For F free $\mathrm{Ext}(F, G) = 0$
- $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, G) \cong G/nG$

Theorem (Universal Coefficient Theorem)**.** *For a chain complex C of free abelian groups we have a split exact sequence*

$$0 \to \mathrm{Ext}(H_{n-1}(C), G) \to H^n(C; G) \xrightarrow{h} \mathrm{Hom}(H_n(C), G) \to 0$$

This is saying that Hom commutes with homology *up to an Ext factor*. This theorem can be applied to make the statement about moving torsion precise. If $T_n(C) \subseteq H_n(C)$ is the torsion subgroup of a finitely generated free abelian group given by homology then

$$H^n(C, \mathbb{Z}) \cong (H_n(C)/T_n(C)) \oplus T_{n-1}(C)$$

Dualising arguments show that for a homology theory the dual will be a cohomology theory.

Hatcher says somethign here about taking coefficients in a field. This would involve taking them not just in an abelian group (Z module) but in a commutative ring (Z algebra).

4 Cup Product

Consider a unital ring R. The singular chain with the free R modules generated on simplicies. Consider singular cohomology with coefficients in R, $H^n(-;R)$. The cup product is a map

$$\smile : C^k(X) \times C^\ell(X) \to C^{k+\ell}(X)$$

$$(\varphi, \psi) \mapsto \varphi \smile \psi$$

where $\varphi \smile \psi$ is defined by linearly extending the map, where $\sigma : \Delta^{k+\ell} \to X$, is a simplex

$$\varphi \smile \psi(\sigma) = \varphi(\sigma|_{[v_0,\dots,v_k]})\psi(\sigma|_{[v_{k+1},\dots,v_\ell]})$$

Recalling that $C^n(X) = \mathrm{Hom}(C_n(X), R)$, and so the value of φ, ψ are in R and can be multiplied there.

Well Defined on Cohomology We want this to be well defined on cohomology, it shouldnt take elements out of the kernel nor the image. There is a simple formula that is helpful here.

what is the relationship with the wedge product on deRham cohomology? Is a product provable aximoatically?

$$\delta(\varphi \smile \psi) = \delta\varphi \smile \psi + (-1)^k \varphi \smile \delta\psi$$

Proof. Some symbol pushing. Just write out the full expression for both.

From this we deduce the closure of the kernel: Let φ, ψ be cocycles. Then we want to show that $\delta(\varphi \smile \psi) = 0$, that is it is still a cocycle. By our formula we have that

$$\delta(\varphi \smile \psi) = \delta\varphi \smile \psi \pm \varphi \smile \partial\psi = 0 \smile \psi \pm \varphi \smile 0 = 0$$

Because we are moding the kernel by the image we also need to check that the image is closed under the action of the kernel. That is we need that the cup of a coboundary with a cocycle is still a coboundary (note that this also checks that the cup of two coboundaries is a coboundary because the image is contained in the kernel). Applying our formular where WLOG φ is a coboundary and ψ is a cocycle gives

$$\delta(\varphi \smile \psi) = \delta\varphi \smile \psi \pm \varphi \smile \delta\psi = \partial\varphi \smile \psi \pm \varphi \smile 0 = \partial\varphi \smile \psi$$

hence the cup of a cocycle and coboundary (RHS) is a coboundary (LHS).

Thus we get an induced map

$$\smile : H^k(X) \times H^\ell(X) \to H^{k+\ell}(X)$$

there is also a relative version for $A, B \subseteq X$ *open*

$$\smile : H^k(X, A) \times H^\ell(X, B) \to H^{k+\ell}(X, A \cup B)$$

If one of A or B was empty then this is the claim that vanishing on A or B is closed under taking cup products. When they are both non-empty we first notice that the cup product on chains is well defined

$$C^k(X, A) \times C^\ell(X, B) \to C^{k+\ell}(X, A + B)$$

where $C^n(X, A + B)$ is the subgroup of $C^n(X)$ given by cochains that vanish on sums of chains in A and B. Then because A and B are open the inclusion

$$C^n(X, A \cup B) \hookrightarrow C^n(X, A + B)$$

induces an iso on homology, giving us the desired cup product.

This then makes

$$H^*(X, A) \quad := \quad \oplus_i H^i(X, A)$$

into a graded ring.

Functoriality

Lemma. *Maps on spaces induce ring homomorphisms:*

$$f : X \to Y \implies f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta).$$

> ***Proof.*** If we have $f : X \to Y$ inducing an R module homomorphism on chains then this is just symbol pushing.

Lemma. *The product is "(anti?)commutative"*

$$\alpha \smile \beta = (-1)^{k\ell} \beta \smile \alpha$$

when coefficients are taken in a commutative ring.

> ***Proof.*** Its quite involved. Uses the proof in homology that homotopic maps are equivilent. Come back?

There is also a relative version of this claim.

Open Problem: All groups are fundamental groups, are all *rings* cohomology rings?

4.1 Geometric Content of the Product

Cup products of nonzero classes occur when the loops intersect.

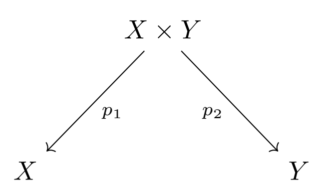
Example.

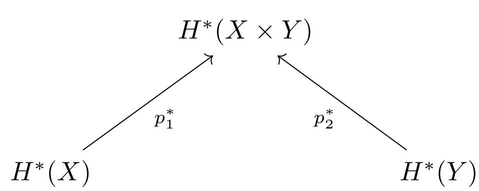
5 Kunneth Formula

There is another type of product, the so called **cross product** (or external cup product), if we take coefficients in R below:

$$H^*(X) \times H^*(Y) \xrightarrow{\times} H^*(X \times Y)$$
$$a \times b = p_X^*(a) \smile p_Y^*(b)$$

where p are the respective projections.





Explicitly we have that

$$a \times b = ap_1 \smile bp_2.$$

This is bilinear, which means it is rarely a homomorphism and so rarely an isomorphism, however it will therefore induce a map

why

$$H^*(X) \otimes_R H^*(Y) \to H^*(X \times Y).$$

This map is a homomorphism of R modules. We can make the tensor product into a ring via

$$(a \otimes b)(c \otimes d) \quad := \quad (-1)^{|b||c|} ac \otimes bd$$

where |x| is the dimension of x, that is the degree of cohomology that it lies in. Then in some cases we have described the ring structure on cohomology.

Theorem. *The cross product $H^*(X) \otimes_R H^*(Y) \to H^*(X \times Y)$ is an isomorphism if for every k, $H^k(Y; R)$ is a finitely generated free R-module.*

This gives us a way to understand the ring structure on the cohomology of the product via understanding the ring structure on the cohomology of the peices.

6 Poincare Duality

Theorem. *Let M be a closed orientable manifold of dimension n, then there is an isomorphism for all k of singular co/homology groups*

$$H_k(M; \mathbb{Z}) \cong H^{n-k}(M; \mathbb{Z})$$

This can be given several generalisations for different classes of manifolds and different rings.

Theorem. *If M is a closed R-orientable n-manifold then $H_k(M; R) \cong H^{n-k}(M; R)$.*

As well as for manifolds with boundary (non-closed)

Theorem. *Let M be a compact R-orientable n-manifold, such that the boundary decomposes as the union of two $(n-1)$-dimensional compact manifolds, called A and B, such that $\partial A = \partial B = A \cap B$. Then there is an isomorphism*

$$H^k(M, A; R) \to H_{n-k}(M, B; R)$$

or for punctured manifolds

Theorem. *If K is a compact, locally contractable subspace of M a closed orientable n-manifold then*

$$H_i(M, M - K; \mathbb{Z}) \cong H^{n-i}(K; \mathbb{Z})$$

Finally for non-compact manifolds we need to define a different cohomology of "compact support" which gives

Theorem. *There is an isomorphism $H_c^k(M; R) \cong H_{n-k}(M; R)$ for M an R-oriented n-manifold.*

There are more generalisations and cases too, involvind different cohomology theories (Cech) too. Apparently this is all just an application of Spanier-Whitehead duality.

6.1 Algebraic Manifold Theory

We will try to present the proof of the following version

Theorem. *If M is a closed R-orientable n-manifold then $H_k(M; R) \cong H^{n-k}(M; R)$.*

so henceforth we will consider only closed (no boundary, compact) manifolds.

6.1.1 Dimension

The dimension of a manifold is given by the *local homology* that is homology relative to everything but a point; for example with integral coefficients

$$\begin{aligned} H_i(M, M - \{x\}) &\cong H_i(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \\ &\cong \tilde{H}_{i-1}(\mathbb{R}^n - \{0\}) \\ &\cong \tilde{H}_{i-1}(S^{n-1}) \\ &= \begin{cases} \mathbb{Z}, & i = \dim(M) \\ 0, & \text{else} \end{cases} \end{aligned}$$

Where the first step we used excision on everything but a local chart of x, then we used that $\mathbb{R}^n$ is contractable to go to reduced homology.

huh?

6.1.2 Orientability

An orientation is intuitively sort of fixing an ordering on a basis? Or its like fixing a north pole, or like fixing an "up"?

what..?

> [Hatcher] Whatever an orientation of $\mathbb{R}^n$ is, it should have the property that it is preserved under rotations and reversed by reflections.

For $\mathbb{R}^2$ the notion of clockwise has this property, for $\mathbb{R}^3$ the notion of right-handed has this property. We define an orientation on $\mathbb{R}^n$ at a point x to be a generator of $H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \cong \mathbb{Z}$, that is the same as fixing an isomorphism with $\mathbb{Z}$.

didnt understand how this satisfies the properties.

If $B \subseteq \mathbb{R}^n$ is a ball containing both x and y then there are canonical isos

$$H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - B) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - \{y\})$$

Thus fixing an orinentation at x fixes one on all of $\mathbb{R}^n$ *via these isomorphisms*. As we earlier remarked by excision we know that $H_n(M, M - \{x\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \cong \mathbb{Z}$ and so we can immediately define an orientation at x in an n-dimensional manifold to be a generator of this group too. We will denote

$$H_n(X|A) \quad := \quad H_n(X, X - A)$$

the local homology of X at A.

We need to make our choice of local orientation consitent across the whole manifold. So we define an **orientation** on an n-dimensional manifold M is an assignment

$$\mu : M \to \cup_x H_n(M|x)$$

sending each x to $\mu_x \in H_n(M|x)$ a local orientation *such that* around every $x \in M$ there is a chart U whose local homology $H_n(M|U)$ has a generator μ_U and moreover for every $y \in U$ μ_y is the image of μ_U under the natural map

$$H_n(M|U) \to H_n(M|y).$$

There is a ball whose homology generator can be taken as the local orientation everywhere in the ball.

If an orientation exists on M we call it **orientable**.

This was for integral coefficients. Can we make sense of the same thing with coefficients in a ring R? An R**-orientation** of M is an assignment

$$\mu : M \to \cup_x H_n(M|x; R) \cong R$$

assigning a generator, or equivilently a unit in R, an element u such that $Ru = R$, at each point, subject to the same local consistency condition.

To prove things about orientations we will reformulate them in terms of coverings and sections.

6.1.3 Orientation as a Section

6.1.4 Comparing R-Orientability and Orientability

Theorem. *An orientable manifold is R-orientable for all R. A non-orientable manifold is R-orientable iff R contains a unit of order 2.*

6.1.5 Orientability as a Condition on Homology

Theorem. *If M is closed connected n-manifold then*

- *If M is R-orientable then for all $x \in M$ the map*

$$H_n(M; R) \to H_n(M|x; R)$$

  is an isomorphism

- *If M is not R-orientable then the map above is injective with image* $\{r \in R : 2r = 0\}$.

Lemma. $H_i(M; R) = 0$ *for* $i > n$.

So in particular $H_n(M; \mathbb{Z})$ is 0 depending on whether M is orientable or not.

6.1.6 Fundamental Class

A fundamental class for M is an element of $H_n(M; R)$ whose image in $H_n(M|x; R)$ is a generator for all x.

Lemma. *M is R-orientable iff M has a fundamental class.*

Proof.

6.1.7 Cap Product

Let X be a space and R be a ring.

$$\frown: C_k(X; R) \times C^\ell(X; R) \to C_{k-\ell}(X; R)$$

$$\sigma \frown \varphi \quad := \quad \varphi(\sigma|_{[v_0,\dots,v_\ell]})\sigma|_{[v_\ell,\dots,v_k]}$$

It is again a symbol pushing exercise to check that this defines a cap product on

$$H_k(X; R) \times H^\ell(X; R) \to H_{k-\ell}(X; R)$$

which is R linear in each variable. There is again a relative version of this cap. There is again a notion in which a map on spaces defines a map on this "product", $f : X \to Y$ then

$$f_*(\alpha) \frown \varphi = f_*\big(\alpha \frown f^*(\varphi)\big)$$

We can relate this to the cup product via

$$\psi(\alpha \frown \varphi) = (\varphi \smile \psi)(\alpha)$$

where ψ, α and φ are on the level of chains (not yet taken co/homology). Thus at least at the level of chains cap and cup are *dual* in the sense that

$$\varphi \smile = (\frown \varphi)^* : Hom((C_\ell; R), R) \to Hom(C_{k+\ell}(X; R), R).$$

At the level of homology this will only be true when Hom commute with taking homology, for example if R is a field.

6.1.8 Cohomology with Compact Support

The strategy of the proof will be to use Mayer-Vietrois. We want some inductive case for the use of Poincare duality for open subsets of our compact manifolds for the Mayer-Vietoris sequence. This is just not true for singular cohomology so we need to introduce a different cohomology that will play the role on the open subsets. This is cohomology with compact support.

Consider the subgroup of the singular cochain groups $C_c^i(X; G)$ consisting of cochains $\varphi : C_i(X) \to G$ such that there exists a compact $K \subseteq X$ on which φ is zero on all chains in $X - K$. Such a cochain is called **compactly supported**. This forms a subcomplex of the singular complex (boundaries preserve compactness). Therefore we can take the cohomology of this subcomplex, which we denote $H_c^*(X; G)$ called **cohomology groups with compact support**. This can also be described as

what is a chain in this space? They mean a linear combination of simplicies where the image of all simplicies is contained in that set?

$$H_c^i(X; G) = \lim_{K \text{ compact}} H^i(X|K; G)$$

Cohomology with compact support is rubbish:

- It is not functorial; only a *proper* map, a map such that the preimage of compact sets is compact, will always give a map on compact cohomology
- Cohomology with compact support is not a homotopy invariant (homotopic spaces have different compact cohomology).

This has convinced me that I can safely ignore them as a technical tool for this proof and not internalise them. What is the *right way* to prove Poincare Duality....

6.2 Proof of Poincare Duality

The full statement of Poincare duality is now: If M is a closed R-orientable n-manifold then caping with a fundamental class $[M]$ defines an isomorphism for all k

$$D_M : H^k(M; R) \to H_{n-k}(M; R),$$

$$\alpha \mapsto [M] \frown \alpha.$$

Note that our M is compact so in particular $H_c^*(M) = H^*(M)$. Actually this proof works for a slightly more general class of manifolds.

Step 1: If $M = U \cup V$ for open U and V, and $D_U, D_V, D_{U \cap V}$ are all isomorphisms then so is D_M.

This uses compact cohomology

Step 2: If M is the union of a sequence of opens $U_1 \subset U_2 \subset \cdots$ and each D_{U_i} is an iso then so is D_M.

This uses compact cohomology

Step 3: Now we prove the case of $M = \mathbb{R}^n$. It is clear that $\mathbb{R}^n \cong \mathring{\Delta}^n$, the interior of the n-simplex. Therefore we have that

$$H^*(\mathbb{R}^n) \cong H^*()$$

Maybe the proof of Poincare duality for DeRham is more informative? Probably not... Does that even make sense?

References